RESULTS CONCERNING THE SCHUTZENBERGER-WALLACE THEOREM

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CHAPTER I

PRELIMINARIES

This initial chapter will present introductory material including an important result which is due to P. Dubreil in its algebraic setting.

<u>1.1 Definition</u>. A topological <u>semigroup</u> S is a nonnull Hausdorff space together with a continuous associative multiplication. Precisely, a semigroup is such a function $m: S \times S \to S$ that

(i) S is a nonnull Hausdorff space,

(ii) m is continuous, and

(iii) m is associative; i.e., for each x,y,z in S,

$$m(x,m(y,z)) = m(m(x,y),z).$$

At times it will be necessary to distinguish between a semigroup and its nontopological counterpart, an <u>algebraic semigroup</u>. It is common usage to say that a semigroup is compact if S is a compact

space and to say that a subset of S is closed if it is closed in a topological sense.

1.2 Definitions. The empty set will be designated by $\square$. If X and Y are subsets of S, then $X^{(-1)}Y = \{w \text{ in } S;\ Xw \cap Y \neq \square\}$, $X^{[-1]}Y = \{w \text{ in } S;\ Xw \subset Y\}$, $YX^{(-1)} = \{w \text{ in } S;\ wX \cap Y \neq \square\}$, and $YX^{[-1]} = \{w \text{ in } S;\ wX \subset Y\}$.

It is noted that if X is a singleton set, then $X^{(-1)}Y = X^{[-1]}Y$ and $YX^{(-1)} = YX^{[-1]}$.

The next result is but a fragment of a result due to A. D. Wallace, another part of which is found in [3].

1.3 Proposition. If Y is closed, then $X^{[-1]}Y$ is closed.

Proof. It is easily verified that $X^{[-1]}Y = \bigcap_{a \in X} a^{(-1)}Y$ and since the function $la\colon S \to S$ defined by $la(s) = as$ is continuous, we have that $(la)^{-1}(Y) = a^{(-1)}Y$ is closed and, consequently, the desired result follows immediately.

<u>1.4 Definition.</u> Letting Y be a subset of S and Δ be the diagonal of $Y \times Y$, then an equivalence relation $\mathfrak{S} \subset Y \times Y$ is a <u>closed congruence</u> on Y if and only if $\Delta\mathfrak{S} \cup \mathfrak{S}\Delta \subset \mathfrak{S}$ and $\mathfrak{S}$ is closed in $Y \times Y$ with respect to the relative topology.

The next two results are well known and are stated here without proof. (The reader may refer to [4] and [6].)

<u>1.5 Proposition.</u> If S is compact or discrete, if $\mathfrak{S}$ is a closed congruence on S and if $\varphi\colon S \to S/\mathfrak{S}$ is the canonical map, then there is a unique continuous function μ such that the diagram

$$\begin{tikzcd} S/\mathfrak{S} \times S/\mathfrak{S} \arrow[r, "\mu"] & S/\mathfrak{S} \\ S \times S \arrow[u, "\varphi \times \varphi"] \arrow[r, "m"'] & S \arrow[u, "\varphi"'] \end{tikzcd}$$

is analytic and thus $S/\mathfrak{S}$ is a semigroup and φ is a continuous homomorphism.

<u>1.6 Theorem</u> (Sierpinski). If $\alpha: A \to B$ and $\theta: A \to C$ are onto functions with the property that $\alpha(a_1) = \alpha(a_2)$ if and only if $\theta(a_1) = \theta(a_2)$, then in the diagram there exist mutually inverse functions f and g such that $g\alpha = \theta$ and $f\theta = \alpha$. Furthermore, if

$$
\begin{tikzcd}
B \arrow[dr, shift left, "g"] & \\
A \arrow[u, "\alpha"] \arrow[r, "\theta"'] & C \arrow[ul, shift left, "f"]
\end{tikzcd}
$$

A, B and C are semigroups and α, θ are morphisms, then f, g are isomorphisms; if, in addition, A is compact, B,C are Hausdorff and α, θ are continuous, then f, g are iseomorphisms.

<u>1.7 Definitions</u>. Throughout this study, and in particular the next theorem, we will make frequent use of the following functions: If $a \in S$ and $B \subset S$ we will define $ra: B \to S$ by $ra(b) = ba$ and $la: B \to S$ by $la(b) = ab$. It is noted that the image of la is $aS \cap Sa$ if $B = a^{(-1)}(Sa)$, because if x is in $aS \cap Sa$, say $x = as = s'a$, then there exists an element in $a^{(-1)}(Sa)$, namely s, such that $x = as$ and, consequently, la maps $a^{(-1)}(Sa)$ onto

$aS \cap Sa$. Moreover, $s' \in (aS)a^{(-1)}$ and $x = s'a$ so that ra maps $(aS)a^{(-1)}$ onto $aS \cap Sa$.

<u>1.8 Theorem</u> (Dubreil). If S is compact or discrete, $a \in S$ and if we define $\mathfrak{E}(a) = \{(x,y);\ x,y \in (aS)a^{(-1)} \text{ and } xa = ya\}$ and $\mathfrak{J}(a) = \{(u,v);\ u,v \in a^{(-1)}(Sa) \text{ and } au = av\}$, then $\mathfrak{E}(a)$ and $\mathfrak{J}(a)$ are congruences on the semigroups $(aS)a^{(-1)}$ and $a^{(-1)}(Sa)$, respectively. If ψ and φ are the appropriate natural homomorphisms in the diagram, then f, g and h are such homeomorphisms that $f\psi = ra$, $g\varphi = la$ and $g^{-1}f = h$; moreover, h is an isomorphism.

$$\begin{tikzcd}
(aS)a^{(-1)}/\mathfrak{E}(a) \arrow[rr, "h"] \arrow[dr, "f"] & & a^{(-1)}(Sa)/\mathfrak{J}(a) \arrow[dl, "g"'] \\
(aS)a^{(-1)} \arrow[u, "\psi"] \arrow[r, "ra"'] & aS \cap Sa & a^{(-1)}(Sa) \arrow[u, "\varphi"'] \arrow[l, "la"]
\end{tikzcd}$$

<u>Proof</u>. The sets $a^{(-1)}(Sa)$ and $(aS)a^{(-1)}$ are nonempty because $a \in a^{(-1)}(Sa) \cap (aS)a^{(-1)}$. The fact that $a^{(-1)}(Sa)$ is closed follows from (1.3), since S is compact and Hausdorff, and it is immediate that $a^{(-1)}(Sa)$ is an algebraic semigroup; in a similar

fashion $(aS)a^{(-1)}$ is a closed semigroup. It is clear that $\mathfrak{I}(a)$ is an equivalence and a right congruence on $a^{(-1)}(Sa)$. If $u,x,y \in a^{(-1)}(Sa)$ and $ax = ay$, then for some $s \in S$, $aux = sax = say = auy$ and, consequently, $\mathfrak{I}(a)$ is also a left congruence. To see that $\mathfrak{I}(a)$ is closed we merely note that $\mathfrak{I}(a) = a^{(-1)}(Sa) \times a^{(-1)}(Sa)) \cap [(1a) \times (1a)]^{-1}(\Delta)$, where Δ is the diagonal of $S \times S$, and that $(1a) \times (1a)$ is continuous. Therefore, in view of (1.5), $a^{(-1)}(Sa)/\mathfrak{I}(a)$ is a semigroup. Clearly, $\varphi(x) = \varphi(y)$ if and only if $1a(x) = 1a(y)$. Since $1a$ maps $a^{(-1)}(Sa)$ onto $aS \cap Sa$, in view of Sierpinski's result we see that such a homeomorphism g exists.

Arguments which are dual to the preceding ones yield the other half of the diagram and, clearly, $h = g^{-1}f$ is a homeomorphism.

If c is in $(aS)a^{(-1)}$, it is easy to see that $f(\psi(c)) = ca$ since $f\psi = ra$ and hence that $g^{-1}f(\psi(c)) = g^{-1}(ca) = g^{-1}(an) = \varphi(n)$ for some $n \in a^{(-1)}(Sa)$, the last equality holding due to the analyticity of the right-hand side of the diagram. Now, if b,c are in $(aS)a^{(-1)}$ and m,n are elements of $a^{(-1)}(Sa)$ such that $ba = am$

and $ca = an$, it follows that $bca = ban = amn$ and therefore $g^{-1}f(\psi(b)\psi(c)) = g^{-1}f(\psi(bc)) = g^{-1}(bca) = g^{-1}(amn) = \varphi(mn) = \varphi(m)\varphi(n) = g^{-1}f(\psi(b))g^{-1}f(\psi(c))$ and, consequently, h is an iseomorphism.

CHAPTER II

EXTENSIONS OF BASTIDA'S RESULTS

The principal purpose of this chapter is to extend, dualize and simplify results given by J. R. Bastida in [1]. The extensions, which culminate in (2.17), are manifold in character and include what is termed the "relative" case (whereas Bastida treats only the "absolute" case, namely, $T = S$) and, in addition, we present results of a non-discrete type which Bastida does not consider. As to the duality, Bastida examines one-half of the possible left-right duality and this chapter indicates that, under certain circumstances, the structures obtained by reversing the multiplication are topologically and algebraically the same. Simplicity is introduced because in the preliminary propositions we isolate those properties of $\mathcal{U}$-slices which are truly necessary for the validity of the arguments. Lastly, it is shown that the Schutzenberger-Wallace Theorem follows as a consequence of these results.

<u>2.1 Definition.</u> If S is a semigroup and A and T are subsets of S, then one defines $L(A,T) = A \cup TA$, $R(A,T) = A \cup AT$ and $H(A,T) = R(A,T) \cap L(A,T)$. When the context clearly indicates which subset T is under consideration, then reference to T is usually omitted, that is, we write $L(A,T) = L(A)$, etc. Moreover, for $T \subset S$, one defines the Relative Green (equivalence) Relations, $\mathcal{L} = \{(x,y);\ L(x) = L(y)\}$, $\mathcal{R} = \{(x,y);\ R(x) = R(y)\}$ and $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$. For $x \in S$, we will let $H_x(T)$ denote the $\mathcal{H}(T)$-class (or slice) containing x; here again reference to T is omitted if the context is clear.

In this chapter, A,B and T will denote subsets of a semigroup S, c will be an element of S and $D = c^{(-1)}A \cap B$.

<u>2.2 Proposition.</u> If $A \subset L(a)$ for all $a \in A$ and $B \subset L(b)$ for all $b \in B$ and if D is nonempty, then $A \subset Sb$ for all $b \in B$.

<u>Proof.</u> If we let $b \in B$ and $d \in D$ and if $b = d$, then $A \subset L(cd) = L(cb) \subset Sb$; if $b \neq d$, then $d \in Tb$ and hence $A \subset L(cd) \subset Sb$.

<u>2.3 Proposition</u>. If X and Y are subsets of S such that $\square \neq X \subset Y$ and if $Y^{(-1)}Y = Y^{[-1]}Y$, then $X^{[-1]}X \subset Y^{[-1]}Y$.

<u>Proof</u>. The hypothesis that X is nonempty is needed to ensure that $X^{[-1]}X \subset X^{(-1)}X$ and so $X \subset Y$ implies that $X^{[-1]}X \subset X^{(-1)}X \subset Y^{(-1)}Y = Y^{[-1]}Y$.

<u>2.4 Proposition</u>. If $y \in S$ such that $y^{(-1)}y \subset D^{[-1]}B \cap A^{[-1]}A$, then $y^{(-1)}y \subset D^{[-1]}D$; moreover, if also $y \in yS$, then $D^{[-1]}D$ is non-empty.

<u>Proof</u>. If $x \in y^{(-1)}y$, then $Dx \subset B$ and $c(Dx) = (cD)x \subset Ax \subset A$ so that $Dx \subset c^{(-1)}A$ and, consequently, $Dx \subset D$, i.e., $x \in D^{[-1]}D$.

The second half of the result follows because $y \in yS$ if and only if $y^{(-1)}y \neq \square$.

<u>2.5 Corollary</u>. If $A \subset L(a)$ for all $a \in A$, if $\square \neq B \subset L(b)$ for all $b \in B$ and if $B^{(-1)}B = B^{[-1]}B$, then $b^{(-1)}b \subset D^{[-1]}D$ for all $b \in B$; moreover, if, in addition, $b \in bS$, then $D^{[-1]}D$ is nonempty.

<u>Proof</u>. We will satisfy the hypothesis of the first part of (2.4): Since $B^{(-1)}B = B^{[-1]}B$, it follows that $b^{(-1)}b \subset b^{(-1)}B \subset$

$B^{(-1)}B = B^{[-1]}B \subset D^{[-1]}B$ and if D is nonempty we have $A \subset Sb$ by (2.2) so that for $a \in A$, $x \in b^{(-1)}b$ we obtain $ax = (sb)x = s(bx) = sb = a$, for some $s \in S$, and, consequently, $b^{(-1)}b \subset A^{[-1]}A$. It is noted that the conclusion also follows if D is empty for then $D^{[-1]}D = S$.

2.6 Proposition. If $a \in A \subset R(a)$ and if $A^{(-1)}a$ is nonempty, then $a \in aS$.

Proof. If $x \in A^{(-1)}a$ we have $a = a'x$ for some $a' \in A$ so that if $a = a'$ the result is immediate and if $a \neq a'$, then $a' = at$ for some $t \in T$ since $A \subset R(a)$ and hence $a = a'x = (at)x = a(tx)$.

It is noted that if A is nonempty the statement $A^{(-1)}a \neq \square$ is implied by the condition $\square \neq a^{(-1)}A \subset \{s \in S; A \subset As\}$, for then $\square \neq \{s \in S; a \in As\} = A^{(-1)}a$.

2.7 Definition. For any $A \subset S$ and $y \in S$, let us define $\mathfrak{S}(A,y) = \{(u,v);\ u,v \in A^{[-1]}A \text{ and } yu = yv\}$ and $\mathfrak{M}(A,y) = \{(u,v);\ u,v \in AA^{[-1]} \text{ and } uy = vy\}$.

__2.8 Proposition.__ If $A^{[-1]}A \neq \square$, then $\mathfrak{S}(A,y)$ is a congruence on $A^{[-1]}A$ if and only if $yu = yv$ implies that $y'u = y'v$ for all $y' \in y(A^{[-1]}A)$.

__Proof.__ For brevity and clarity, let $\mathfrak{S} = \mathfrak{S}(A,y)$ in this proof. If $\mathfrak{S}$ is a congruence on $A^{[-1]}A$, then $(\Delta\mathfrak{S} \cup \mathfrak{S}\Delta) \subset \mathfrak{S}$ where Δ is the diagonal of $A^{[-1]}A \times A^{[-1]}A$ and thus, letting $w \in A^{[-1]}A$ and $(u,v) \in \mathfrak{S}$, we have $ywu = ywv$ so that $y'u = y'v$ for all y' in $y(A^{[-1]}A)$.

Conversely, if the condition holds, namely, $yu = yv$ implies that $y'u = y'v$ for all y' in $y(A^{[-1]}A)$ and if $w \in A^{[-1]}A$, $(u,v) \in \mathfrak{S}$, then the condition implies that $(yw)u = (yw)v$ because yw is in $y(A^{[-1]}A)$ and therefore $\mathfrak{S}$ is a left congruence on $A^{[-1]}A$. It is evident that $\mathfrak{S}$ is a right congruence.

__2.9 Corollary.__ If b is in $B \cap bS$, if $A \subset L(x)$ for all x in A, if $\square \neq B \subset L(x)$ for all x in B, if $B^{(-1)}B = B^{[-1]}B$ and if D is nonempty, then $\mathfrak{S}(D,b)$ is a congruence on $D^{[-1]}D$ and $\mathfrak{S}(D,b) = \mathfrak{S}(D,b')$ for $b' \in B$.

Proof. In view of (2.5) the hypothesis implies that $D^{[-1]}D$ is nonempty so that in order to prove that $\mathfrak{S}(D,b)$ is a congruence on $D^{[-1]}D$ it suffices to show that the condition of (2.8) is satisfied. In the case that B is a singleton set, say $B = \{b\}$, we have from (2.3) that $b(D^{[-1]}D) \subset b(B^{[-1]}B) \subset B = \{b\}$ so that the condition is trivially satisfied. If card $B > 1$, then it follows that b is in Sb and thus that $B \subset Sb$. Letting $(u,v) \in \mathfrak{S}(D,b)$ and $b' \in b(D^{[-1]}D)$ we have $b' \in b(b^{(-1)}B) \subset B \subset Sb$ since $D^{[-1]}D \subset b^{(-1)}B$ so that if $b' = sb$ we obtain $b'u = (sb)u = s(bu) = s(bv) = (sb)v = b'v$ and therefore the condition of (2.8) holds.

Momentarily fixing distinct elements b and b' in B and letting (u,v) be an element of $\mathfrak{S}(D,b)$, we use the fact that $B \subset L(b)$ to obtain $b'u = (tb)u = t(bu) = t(bv) = (tb)v = b'v$, where $t \in T$, so that (u,v) is also in $\mathfrak{S}(D,b')$. Clearly, in a similar fashion we have $\mathfrak{S}(D,b') \subset \mathfrak{S}(D,b)$.

2.10 Proposition. $A^{[-1]}A \subset (xA)^{[-1]}(xA)$ for all $x \in S$.

Proof. If $y \in A^{[-1]}A$, then $(xA)y = x(Ay) \subset xA$ so that y is in $(xA)^{[-1]}(xA)$.

2.11 Proposition. For any elements x,y and $z \in S$, if $x \in Sy$, then $\mathfrak{S}(A,y) \subset \mathfrak{S}(zA,x)$.

Proof. If $(u,v) \in \mathfrak{S}(A,y)$, then in view of (2.10) it remains only to verify that $xu = xv$: Letting $x = sy$ we have $s(yu) = (sy)u = xu$ and in a similar manner $s(yv) = xv$ so that $xu = xv$.

2.12 Corollary. If $A \subset L(a)$ for all a in A, if $B \subset L(b)$ for all $b \in B$ and if D is nonempty, then $\mathfrak{S}(D,b) \subset \mathfrak{S}(cD,a)$ where $a \in A$ and $b \in B$.

Proof. The hypothesis is that of (2.2) so that $A \subset Sb$ and therefore in view of (2.11) the conclusion is evident.

2.13 Proposition. If A, B and D are nonempty sets such that $A \subset L(a)$ for all $a \in A$ and $B \subset L(b)$ for all $b \in B$, if $A^{(-1)}A = A^{[-1]}A$ and $B^{(-1)}B = B^{[-1]}B$ and if $b \in bS$ for some $b \in B$, then $\mathfrak{S}(cD,a)$ is a congruence on $(cD)^{[-1]}(cD)$ for any $a \in A$.

Proof. In view of (2.5) we have $D^{[-1]}D \neq \square$ and so $(cD)^{[-1]}(cD)$ is nonempty since $D^{[-1]}D \subset (cD)^{[-1]}(cD)$. We will now verify that the condition of (2.8) is satisfied: In the case that $A = \{a\}$ it follows from (2.3) that $a[(cD)^{[-1]}(cD)] = \{a\}$ and as a result the condition is trivially fulfilled.

If card $A > 1$, then we obtain $a \in Sa$ and, therefore, $A \subset Sa$. Letting $(u,v) \in \mathfrak{S}(cD,a)$ and $a' \in a[(cD)^{[-1]}(cD)]$ we have a' in $a(A^{[-1]}A) \subset Sa$ since $(cD)^{[-1]}(cD) \subset A^{[-1]}A$ by (2.3). Consequently, if $a' = sa$, then we see that $a'u = sau = sav = a'v$ and thus the condition of (2.8) holds.

The next result is well known and it is due to B. J. Pettis; consequently, its proof is omitted.

2.14 Proposition. If a compact semigroup is algebraically a group, then it is a topological group.

2.15 Proposition (Induced Homomorphism Theorem). If α and β are congruences on semigroups A and B, respectively, such that $A \subset B$

and $\alpha \subset \beta$, then in the diagram, where f and g are the appropriate canonical maps and i is the inclusion map, there exists a homomorphism i^* such that the diagram is analytic.

$$\begin{tikzcd} A/\alpha \arrow[r, "i^*"] & B/\beta \\ A \arrow[u, "f"] \arrow[r, "i"'] & B \arrow[u, "g"'] \end{tikzcd}$$

<u>Proof</u>. This proposition follows from an evident extension of the version of the Induced Homomorphism Theorem given in [2] and its corollary there.

<u>2.16 Lemma</u>. If $A^{(-1)}A \subset A^{[-1]}A$ and if $B^{(-1)}B \subset B^{[-1]}B$, then $D^{(-1)}D \subset D^{[-1]}D$ and, consequently, $D^{[-1]}D = d^{(-1)}D$ for all d in D provided that D is nonempty.

<u>Proof</u>. First of all, it is clear that $D^{(-1)}D \subset B^{(-1)}B \subset B^{[-1]}B$. Since we may assume that $D^{(-1)}D$ is nonempty, say $t \in D^{(-1)}D$, then, using the fact that $cD \subset A$, we have that $(At \cap A) \supset (cDt \cap cD) \supset c(Dt \cap D) \neq \Box$. Again noting that $cD \subset A$, it follows that $cDt \subset At \subset A$ so that $Dt \subset c^{(-1)}A$. As a result, $Dt \subset c^{(-1)}A \cap B = D$ and so $t \in D^{[-1]}D$.

The second conclusion follows from the fact that $D^{[-1]}D =$
$\bigcap \{d^{(-1)}D;\ d \in D\} \subset d^{(-1)}D \subset \bigcup \{d^{(-1)}D;\ d \in D\} = D^{(-1)}D.$

<u>2.17 Theorem</u>. (a) Let S be compact or discrete and A and B be nonempty sets satisfying these three conditions:

(i) $A \subset L(a)$ for all a in A,

(ii) $B \subset L(b) \cap R(b)$ for all $b \in B$,

(iii) $A^{(-1)}A = A^{[-1]}A$ and $B^{(-1)}B = B^{[-1]}B$.

If card $B > 1$ and if D is both nonempty and closed, then this diagram is analytic:

$$\begin{tikzcd}
& D^{[-1]}D/\mathfrak{S}(D,d) \arrow[dl, "g"'] \\
D & D^{[-1]}D \arrow[l, "\mathrm{id}"] \arrow[u, "\varphi"']
\end{tikzcd}$$

where $D^{[-1]}D/\mathfrak{S}(D,d)$, for d in D, is a topological group, φ is the canonical map and g is a homeomorphism.

(b) If, in addition, card $A > 1$, the preceding analytic diagram may be extended:

$$
\begin{tikzcd}
D^{[-1]}D \arrow[rr, "1d"] \arrow[dr, "\varphi"] \arrow[ddd, phantom, "\cap"] & & D \arrow[ddd, "1c"] \\
& D^{[-1]}D/\mathfrak{S}(D,d) \arrow[ur, "g"'] \arrow[d, "\gamma"] & \\
& (cD)^{[-1]}(cD)/\mathfrak{S}(cD,cd) \arrow[dr, "m"] & \\
(cD)^{[-1]}(cD) \arrow[rr, "1(cd)"'] \arrow[ur, "\alpha"] & & cD
\end{tikzcd}
$$

where $(cD)^{[-1]}(cD)/\mathfrak{S}(cD,cd)$ is a topological group, α is a canonical map, m is a homeomorphism and γ is a continuous epimorphism.

Proof. We will consider only the case where S is compact since the situation where S is discrete follows in a similar manner with the topological results omitted.

Since card $B > 1$ and $B \subset R(b)$ for all b in B, it follows that $d \in dS$ for each $d \in D$ and so, by (2.5), $D^{[-1]}D$ is nonempty. Using (1.3), $D^{[-1]}D$ is closed and compact because D is closed and S is compact. It is immediate from (2.9) that $\mathfrak{S}(D,d)$ is a congruence on

$D^{[-1]}D$ for any $d \in D$ and, moreover, $\mathfrak{S}(D,d)$ is closed because it is easily verified that $\mathfrak{S}(D,d) = (D^{[-1]}D \times D^{[-1]}D) \cap (1d \times 1d)^{-1}(\Delta)$ and so $D^{[-1]}D/\mathfrak{S}(D,d)$ is a compact topological semigroup by (1.5). With (2.14) in mind we proceed to show that $D^{[-1]}D/\mathfrak{S}(D,d)$ is algebraically a group:

We may select an element q of $d^{(-1)}d$ because $d^{(-1)}d$ is nonempty if and only if d is in dS. Then, since $D \subset L(d)$ we have $dx \in L(d)$ for $x \in D^{[-1]}D$ and so it is easily verified that $dx = dxq$. Since $D^{[-1]}D$ is a semigroup we have $(x,xq) \in \mathfrak{S}(D,d)$ and thus $\varphi(x) = \varphi(x)\varphi(q)$ so that $\varphi(q)$ is a right unit for $\varphi(x)$. If $d = dx$ we have $d = dxx$ and if $d \neq dx$, then $d = dxt$ for some $t \in T$ because $B \subset R(b)$ for all $b \in B$ so that in either case there is an element x' such that $d = dxx'$ and, consequently, $dq = dxx'$. In view of (2.16), x' is in $D^{[-1]}D$ and therefore $\varphi(q) = \varphi(x)\varphi(x')$ indicating that $\varphi(x)$ has a right inverse.

In order to show that $D \stackrel{t}{=} D^{[-1]}D/\mathfrak{S}(D,d)$ we will make use of Dubreil's result, that is, (1.8): Card $B > 1$ implies that

$B \subset L(d) \subset Sd$ and thus using (2.3) we have $D^{[-1]}D \subset d^{(-1)}B \subset d^{(-1)}(Sd)$. The fact that $D^{(-1)}D \subset D^{[-1]}D$ implies that the restriction of ld to $D^{[-1]}D$ has as its image D, for if $d' \in D$ and $d' = d$ we know that $d^{(-1)}d$ is nonempty and if $d' \neq d$, $d' = dt$ for some $t \in T$ so that in either case $d' = dt'$ and thus $t' \in D^{(-1)}D \subset D^{[-1]}D$; the other inclusion is clear because $D^{[-1]}D \subset d^{(-1)}D$ implies that $d(D^{[-1]}D) \subset D$. Therefore, $\varphi(D^{[-1]}D) \stackrel{t}{=} D$ because $g\varphi = ld$ where g is the appropriate homeomorphism of (1.8).

To prove the second part of the theorem we begin by proving the existence of a continuous epimorphism γ. Since $D^{[-1]}D \subset (cD)^{[-1]}(cD)$ and $\mathcal{S}(D,d) \subset \mathcal{S}(cD,cd)$, the Induced Homomorphism Theorem gives us the existence of a function γ such that $\gamma\varphi = \alpha i$ where i is the inclusion map. φ is closed because it is continuous, $D^{[-1]}D$ is compact and $D^{[-1]}D/\mathcal{S}(D,d)$ is Hausdorff from a result in [5] and thus since it is also true that φ is an onto function and $\alpha i = \gamma\varphi$ is continuous, we have that γ is continuous from another result in [5]. It is clear that γ is a homomorphism and so it remains to verify that it is

an onto function: If Y is an element of $(cD)^{[-1]}(cD)/\mathfrak{S}(cD,cd)$ and y is in Y, then $cdy = cd'$ for some $d' \in D$. If $d = d'$ and q is in $d^{(-1)}d$, then $cdy = cd = cdq$ and it follows that (y,q) is in $\mathfrak{S}(cD,cd)$ so that $\gamma(\varphi(q)) = Y$. If $d \neq d'$, then for some $t \in T$ we see that $d' = dt$ because $B \subset R(d)$ and we note that $t \in D^{(-1)}D = D^{[-1]}D$. Then, since $cdy = cdt$, it is true that (y,t) is in $\mathfrak{S}(cD,cd)$ and it follows that $\gamma(\varphi(t)) = Y$. We conclude, therefore, that γ is an onto function.

It is next noted that $(cD)^{[-1]}(cD)$ is nonempty because $D^{[-1]}D \subset (cD)^{[-1]}(cD)$ and that in an analogous manner to the proof of the first part of the theorem it is easy to verify that $(cD)^{[-1]}(cD)$ is closed and compact, that $\mathfrak{S}(cD,cd)$ is a closed congruence on $(cD)^{[-1]}(cD)$ and, hence, that $(cD)^{[-1]}(cD)/\mathfrak{S}(cD,cd)$ is a topological semigroup. Then, since $D^{[-1]}D/\mathfrak{S}(D,d)$ is a group and γ is an epimorphism, it follows that $(cD)^{[-1]}(cD)/\mathfrak{S}(cD,cd)$ is a topological group.

As in the proof of the first part of the theorem, we will use Dubreil's result in order to obtain $cD \stackrel{t}{=} (cD)^{[-1]}(cD)/\mathfrak{S}(cD,cd)$:

Card $A > 1$ implies that $A \subset L(cd) \subset Scd$ and thus using (2.3) we find that $(cD)^{[-1]}(cD) \subset (cd)^{(-1)}A \subset (cd)^{(-1)}(Scd)$. If $d' \in D$ and if $d' = d$, then $(cd)^{(-1)}(cd)$ is nonempty since $d^{(-1)}d$ is non-empty and if $d' \neq d$, then $d' = dt$ for some $t \in T$ and hence $cd' = cdt$ so that in either case $cd' = cdt'$ and t' is in $D^{(-1)}D \subset D^{[-1]}D \subset (cD)^{[-1]}(cD)$; the other inclusion is clear because $(cD)^{[-1]}(cD) \subset (cd)^{(-1)}(cD)$ implies that $cd[(cD)^{[-1]}(cD)] \subset cD$. Therefore, $\varphi[(cD)^{[-1]}(cD)]$ is homeomorphic to cD since $m\varphi = 1(cd)$ where m is the appropriate homeomorphism, namely, g, of (1.8).

<u>2.18 Proposition</u>. Under the hypotheses of (2.17), if T is a subsemigroup, then B, D and cD are contained in $\mathcal{H}$-slices.

<u>Proof</u>. $T^2 \subset T$ implies that $\mathcal{H} = \{(x,y) \in S \times S;\ H(x) = H(y)\}$ and so for $b \in B$, $H_b = \{x \in S;\ H(x) = H(b)\}$. It then follows that, since $B \subset R(b) \cap L(b) = H(b)$ for $b \in B$, we have $B \subset H_b$. Clearly, $D \subset H_b$ because $D \subset B$.

Next we notice that $cD \subset A \subset L(a)$ for $a \in A$ and, in particular, $cD \subset L(cd)$ for $d \in D$. Also, we find that

$cD \subset cB \subset cR(b) = R(cb)$ for $b \in B$ so that $cD \subset R(cd)$ for $d \in D$. Consequently, $cD \subset L(cd) \cap R(cd) = H(cd)$ for $d \in D$ and it follows easily that $cD \subset H_{cd}$.

<u>2.19 Theorem</u>. Suppose S is compact or discrete and let us define $K = Hw^{(-1)} \cap J$ where $w \in S$ and where H and J are nonempty sets in S satisfying these three conditions:

(i) $H \subset R(h)$ for all $h \in H$,

(ii) $J \subset R(x) \cap L(x)$ for all $x \in J$,

(iii) $HH^{(-1)} = HH^{[-1]}$ and $JJ^{(-1)} = JJ^{[-1]}$.

If card $J > 1$ and if K is both nonempty and closed, then this diagram is analytic:

$$\begin{tikzcd} KK^{[-1]}/\mathfrak{M}(K,k) \arrow[dr, "f"] & \\ KK^{[-1]} \arrow[u, "\psi"] \arrow[r, "rk"'] & K \end{tikzcd}$$

where $KK^{[-1]}/\mathfrak{M}(K,k)$, $k \in K$, is a topological group, ψ is the canonical map and f is a homeomorphism.

Proof. All the results preceding (2.17) may be easily "dualized" so that this theorem may be proved in a manner analogous to the proof of (2.17).

2.20 Theorem. If the hypotheses of part (a) of (2.17) and (2.19) hold and if d is in $D \cap K$, then we naturally speak of the results of (2.19) as being the "mirror image" of the results in part (a) of (2.17) in view of this analytic extension of Dubreil's diagram,

$$
\begin{tikzcd}
KK^{[-1]}/\mathfrak{M} \arrow[r, phantom, "\subset"] & (dS)d^{(-1)}/\mathfrak{E} \arrow[rr, "h"] \arrow[dr, "f"] & & d^{(-1)}(Sd)/\mathfrak{J} \arrow[r, phantom, "\supset"] \arrow[dl, "g"'] & D^{[-1]}D/\mathfrak{S} \\
KK^{[-1]} \arrow[u, "\psi'"] \arrow[r, phantom, "\subset"] & (dS)d^{(-1)} \arrow[u, "\psi"] \arrow[r, "rd"'] & dS \cap Sd & d^{(-1)}(Sd) \arrow[u, "\varphi"'] \arrow[l, "ld"] \arrow[r, phantom, "\supset"] & D^{[-1]}D \arrow[u, "\varphi'"']
\end{tikzcd}
$$

where for brevity $\mathfrak{M} = \mathfrak{M}(K,d)$, $\mathfrak{J} = \mathfrak{J}(d)$, $\mathfrak{E} = \mathfrak{E}(d)$ and $\mathfrak{S} = \mathfrak{S}(D,d)$ and ψ' is the restriction of ψ to $KK^{[-1]}/\mathfrak{M}$ and similarly for φ'. Moreover, if $D = K$, then in the diagram the restriction of h to $KK^{[-1]}/\mathfrak{M}$ is an iseomorphism with image $D^{[-1]}D/\mathfrak{S}$.

Proof. The first part of this theorem follows easily from the results of (1.8), (2.17) and (2.19). In addition, from (2.19) we see

that the restriction of f to $KK^{[-1]}/\mathfrak{M}$ is K and from (2.17) we find that $g^{-1}(D) = D^{[-1]}D/\mathfrak{S}$ and from Dubreil's result we recall that $h = g^{-1}f$ is an iseomorphism, so that putting these remarks together the conclusion follows because $h(KK^{[-1]}/\mathfrak{M}) = g^{-1}f(KK^{[-1]}/\mathfrak{M}) = g^{-1}(K) = g^{-1}(D) = D^{[-1]}D/\mathfrak{S}$.

In view of its position in the diagram, an iseomorphism such as that expressed in (2.20) is known as "turning the corner."

The next theorem has been presented in its algebraic context for $T = S$ in [1] and it formed the cornerstone of that work. It is in view of this last remark that we attach the author's name to the theorem in its presentation. This result is also important to me because it served as the prime motivation of this dissertation. Subsequent to this theorem it will be shown that a well-known theorem, due originally to M. P. Schutzenberger and to A. D. Wallace in its present formulation, follows in part as a corollary.

<u>2.21 Lemma</u>. $H_w^{(-1)}H_w = H_w^{[-1]}H_w$ for w in S.

Proof. The reader may find the proof of this result in [4] where it is shown to be a consequence of Green's Lemma.

2.22 Theorem (Bastida). Let S be compact or discrete, T be a closed subset of S and $D = c^{(-1)}H_x \cap H_y$. If card $H_x > 1$, card $H_y > 1$ and D is nonempty, then this diagram is analytic:

$$\begin{tikzcd}
D^{[-1]}D \arrow[rr, "1d"] \arrow[dr, "\varphi"'] \arrow[ddd, phantom, "\cap"] & & D \arrow[ddd, "1c"] \\
& D^{[-1]}D/\mathfrak{S}(D,d) \arrow[ur, "g"'] \arrow[d, "\gamma"] & \\
& (cD)^{[-1]}(cD)/\mathfrak{S}(cD,cd) \arrow[dr, "m"] & \\
(cD)^{[-1]}(cD) \arrow[ur, "\alpha"] \arrow[rr, "1(cd)"'] & & cD
\end{tikzcd}$$

where d is in D, $D^{[-1]}D/\mathfrak{S}(D,d)$ and $(cD)^{[-1]}(cD)/\mathfrak{S}(cD,cd)$ are topological groups, φ and α are canonical maps, g and m are homeomorphisms and γ is a continuous epimorphism.

Proof. We will easily verify that the hypotheses of (2.17) are fulfilled, where H_x and H_y will be A and B, respectively. In view of (2.21) and because H_x and H_y are $\mathcal{H}$-slices, we have that (i), (ii) and (iii) of (2.17) hold. If S is compact and T is closed, then H_x and H_y are closed so that $c^{(-1)}H_x$ is closed and, consequently, D is closed. Therefore, it may be seen that all the hypotheses of (2.17) are satisfied and hence this proposition now follows as an immediate corollary.

2.23 Theorem (Schutzenberger-Wallace). If S is compact or discrete, if T is a closed subset of S and if y is an element of S such that card $H_y > 1$, then H_y is homeomorphic to the topological group, $y^{(-1)}H_y/\mathfrak{S}(H_y,y)$, and the groups $y^{(-1)}H_y/\mathfrak{S}(H_y,y)$ and $H_y y^{(-1)}/\mathfrak{M}(H_y,y)$ are isomorphic.

Proof. Using the dual of (2.21) we see that card $H_y > 1$ implies that $H_y H_y^{[-1]}$ is nonempty so that letting $H_y = H_x$ in (2.22) and c be an element of $H_y H_y^{[-1]}$, we have $D = c^{(-1)}H_y \cap H_y = H_y$ because $H_y \subset c^{(-1)}H_y$. The first part of this theorem now follows as a corollary to (2.22) since we have that $y^{(-1)}H_y = H_y^{[-1]}H_y$ from [4].

In a similar manner we may choose an element w in $H_y{}^{[-1]}H_y$ so that the set K of (2.19) and (2.20) is H_y. Therefore, by (2.20), we may turn the corner and find that $y^{(-1)}H_y/\mathfrak{C}(H_y,y)$ and $H_y y^{(-1)}/\mathfrak{M}(H_y,y)$ are iseomorphic.

CHAPTER III

A CHARACTERIZATION OF THE CLOSED SUBGROUPS OF THE SCHUTZENBERGER GROUP

<u>3.1 Proposition.</u> If b is in S and A is a subset of S, then with regard to the statements

(1) $b^{(-1)}A$ is a semigroup,

(2) $b^{(-1)}A \subset [b(b^{(-1)}A)]^{[-1]}A$,

(3) $b^{(-1)}A \subset A^{[-1]}A$,

(4) $b^{(-1)}A \subset \{x \in S;\ Ax = A\}$,

(5) $A = Cb$ and $C^2b \subset Cb$ where $C \subset S$

the dependency is indicated by the diagram,

$$\begin{tikzcd}
(4) \arrow[dr] & & & \\
& (3) \arrow[r] & (2) \arrow[r, leftrightarrow] & (1) . \\
(5) \arrow[ur] & & &
\end{tikzcd}$$

Moreover, if $A \subset bS$, then (1) implies (3); consequently, if $b \in A$ and $b^{(-1)}A$ is a semigroup, then $b^{(-1)}A = A^{[-1]}A$.

Proof. If x and y are elements of $b^{(-1)}A$, then condition (2) implies that $b(xy) = (bx)y \in A$ so that $b^{(-1)}A$ is a semigroup; conversely, if x and y are elements of a semigroup $b^{(-1)}A$, it then follows that $b(xy) = (bx)y \in [b(b^{(-1)}A)]y \subset A$ and so $y \in [b(b^{(-1)}A)]^{[-1]}A$. It is easy to see that $b^{(-1)}A \subset A^{[-1]}A$ implies the validity of condition (2) because $b(b^{(-1)}A)$ is a subset of A and, since it is always the case that $\{x;\ Ax = A\} \subset A^{[-1]}A$, it is clear that condition (4) implies condition (3). If condition (5) holds and $x \in b^{(-1)}A$, then $(Cb)x = C(bx) \subset CA = C(Cb) \subset Cb$ so that x is in $A^{[-1]}A$ and condition (4) is satisfied.

$b^{(-1)}A$ is a semigroup means that $(b^{(-1)}A)(b^{(-1)}A) \subset b^{(-1)}A$ so that, multiplying by b on the left and using the fact that $A \subset bS$, we obtain $A(b^{(-1)}A) \subset A$ and, consequently, $b^{(-1)}A \subset A^{[-1]}A$. If, in addition, b is in A, then it is clear that $b^{(-1)}A = A^{[-1]}A$.

It is possible to indicate, as shown by the following examples, that the implications among conditions (2) through (5) of (3.1) may not be reversed:

<u>3.2 Examples.</u> (a) Consider the semigroup S defined by the multiplication table,

	0	1	2
0	0	0	0
1	0	0	1
2	0	1	2

and let $b = 1$ and $A = \{0,2\}$.

Then $b^{(-1)}A = \{0,1\}$ which is clearly a subsemigroup so that condition (2) holds and yet $b^{(-1)}A$ is not a subset of $A^{[-1]}A$ which is $\{0,2\}$.

(b) Using the semigroup S defined in (a) and letting $b = 1$, $A = \{0,1\}$ and $C = \{0,3\}$ we have that $A = Cb$ and $C^2b = Cb$ whereas $b^{(-1)}A = S$ and $\{x \in S;\ Ax = A\} = \{2\}$ so that neither condition (3) nor condition (5) implies condition (4).

(c) If we let a semigroup S be defined by the table

	0	1	2	3
0	0	0	2	2
1	1	1	3	3
2	2	2	0	0
3	3	3	1	1

and if $b = 1$ and $A = \{0,1,2\}$, then $\{x \in S;\ Ax = A\} =$ $b^{(-1)}A = \{0,1\}$; however, the only set C such that $A = Cb$ is A itself and, in this case, we find that $C^2b = S$. Consequently, neither condition (3) nor condition (4) implies condition (5).

3.3 Theorem. If S is such a semigroup of a compact group that S is either open or closed, then S is a closed subgroup.

Proof. This result is well known and the reader is referred to [7].

3.4 Theorem. Let S be a compact or discrete semigroup, T be a closed subset of S and y be such an element of S that card $H_y > 1$. If G is a closed topological subgroup of the Schutzenberger group, $y^{(-1)}H_y/\mathcal{B}(H_y,y)$, and if w is an element of H_y, then, letting $A_w = w(\varphi^{-1}G)$ where φ is the homomorphism of Dubreil's diagram, it is true that $\varphi^{-1}G = A_w^{[-1]}A_w = A_w^{(-1)}A_w$ and A_w is topologically equivalent to G. Conversely, if A is a nonempty closed subset of H_y such that $A^{[-1]}A = A^{(-1)}A$, then for a in A the following diagram is analytic and, as a result, $A \stackrel{t}{=} \varphi(A^{[-1]}A)$ which is a subgroup of the Schutzenberger group:

$$\begin{tikzcd}
 & & & a^{(-1)}A/\mathfrak{S}(A,a) \arrow[dddddlll, "g''"'] & & & \\
 & & & \cap & & & \\
 & & & a^{(-1)}H_a/\mathfrak{S}(H_a,a) \arrow[ddddll, "g'"'] & & & \\
 & & & \cap & & & \\
 & & & a^{(-1)}(Sa)/\mathfrak{J}(a) \arrow[dl, "g"'] & & & \\
A \arrow[r, phantom, "\subset"] & H_a \arrow[r, phantom, "\subset"] & aS \cap Sa & & a^{(-1)}(Sa) \arrow[ll, "1a"] \arrow[ul, "\varphi"'] \arrow[r, phantom, "\supset"] & a^{(-1)}H_a \arrow[uuull, "\varphi'"'] \arrow[r, phantom, "\supset"] & A^{[-1]}A \arrow[uuuuulll, "\varphi''"']
\end{tikzcd}$$

where the primes and double primes indicate that those functions are restrictions of φ and g.

Proof. We will consider only the case where S is compact because the situation where S is discrete follows in a similar manner with the topological results omitted.

If G is contained in the Schutzenberger group, namely, $\varphi(w^{(-1)}H_y)$, for any w in H_y, then $\varphi^{-1}G \subset w^{(-1)}H_y$, for if x is in $w^{(-1)}(Sw)$ and $\varphi(x) \in G$, then $\varphi(x) = \varphi(y)$ for some y in $w^{(-1)}H_y$ so that $wx = wy \in H_y$ and $x \in w^{(-1)}H_y$; moreover, $\varphi^{-1}G = w^{(-1)}[w(\varphi^{-1}G)]$ because if x is in $w^{(-1)}[w(\varphi^{-1}G)]$ we have $wx \in w(\varphi^{-1}G)$ and $\varphi(x) \in \varphi(\varphi^{-1}G) = G$ so that $x \in \varphi^{-1}G$ and the reverse

set inclusion is clear. Letting $A_w = w(\varphi^{-1}G)$ we have that $\varphi^{-1}G = w^{(-1)}A_w$ and so, since $\varphi^{-1}G$ is a semigroup, we may use (3.1) to obtain $\varphi^{-1}G \subset [w(w^{(-1)}A_w)]^{[-1]}A_w = A_w^{[-1]}A_w$. Since $w^{(-1)}w$ is non-empty, it follows that $\varphi(w^{(-1)}w)$ is the identity of the Schutzenberger group, as may readily be seen by a proper modification of the proof of (2.17), so that $\varphi^{-1}G$ contains an element q such that wq = w. Consequently, $w \in A_w$ and as a result $\varphi^{-1}G = A_w^{[-1]}A_w$. Because $A_w \subset H_w \subset R(w)$, the restriction of the lw function of Dubreil's diagram to $A_w^{[-1]}A_w$ has as its image A_w and therefore, by Dubreil's result, $G = \varphi(A_w^{[-1]}A_w)$ is homeomorphic to A_w. It may be noted that A_w is closed because $w^{(-1)}H_y$ is compact. If $x \in \varphi^{-1}G$ and $a \in A_w$, then card $H_y > 1$ implies that $wx = atx = wt'tx$ where $t \in T \cap w^{(-1)}H_y$ because $H_y^{(-1)}H_y = w^{(-1)}H_y$ and $t' \in T \cap \varphi^{-1}G$ because $\varphi^{-1}G = w^{(-1)}A_w$. Therefore, $\varphi(x) = \varphi(t'tx) = \varphi(t')\varphi(t)\varphi(x)$ and since $\varphi(t')$ and $\varphi(x)$ have group inverses in G, it follows, letting $\varphi(t')^{-1}$ be the inverse of $\varphi(t')$, that $\varphi(t')^{-1} = \varphi(t) \in G$ and so $t \in \varphi^{-1}G$ and, consequently, $w(\varphi^{-1}G) \subset a(\varphi^{-1}G)$; moreover, $ax = wt'x$ and so we have the reverse set

inclusion, namely, $a(\varphi^{-1}G) \subset w(\varphi^{-1}G)$. As a result, for each $a \in A_w$, it follows that $\varphi^{-1}G = a^{(-1)}[a(\varphi^{-1}G)] = a^{(-1)}[w(\varphi^{-1}G)] = a^{(-1)}A_w$ and, consequently, $A_w^{[-1]}A_w = A_w^{(-1)}A_w$.

Conversely, if A is a nonempty closed subset of H_y such that $A^{[-1]}A = A^{(-1)}A$, then, for $a \in A$, $A^{[-1]}A = a^{(-1)}A \subset a^{(-1)}H_y \subset a^{(-1)}(Sa)$, the last inclusion being true because card $H_y > 1$, so that $\varphi(A^{[-1]}A) \subset \varphi(a^{(-1)}H_y)$, the Schutzenberger group, where φ is the appropriate canonical map in Dubreil's result. Since A is closed, in view of (3.3), we have that $\varphi(A^{[-1]}A)$ is a group. (In the discrete case, a somewhat longer argument, similar to that used in the proof of (2.17), shows that $\varphi(A^{[-1]}A)$ is a group.) Lastly, since $A^{[-1]}A = a^{(-1)}A$ and $A \subset R(a)$, the restriction of the la function of Dubreil's diagram to $A^{[-1]}A$ has as its image A and, therefore, using Dubreil's result, $\varphi(A^{[-1]}A)$ is homeomorphic to A.

<u>3.5 Proposition.</u> If A and B are subsets of S such that A is nonvoid and connected and B is a component of S, then $A^{[-1]}B = A^{(-1)}B$.

Proof. If y is an element of $A^{(-1)}B$, then Ay is connected and so it follows that $B \cup A(A^{(-1)}B)$ is connected. Then, since B is a component, $A(A^{(-1)}B) \subset B$ and we have that $A^{(-1)}B \subset A^{[-1]}B$. The fact that A is nonempty is used to ensure that $A^{[-1]}B \subset A^{(-1)}B$.

3.6 Corollary. If A is a component of S, then $A^{(-1)}A = A^{[-1]}A$. Consequently, if S is compact or discrete, then for a set A contained in an $\mathcal{K}$-slice having cardinality > 1 to be a component of S it is necessary that A be homeomorphic to the topological group, $\varphi(A^{[-1]}A)$, where φ is the canonical map of Dubreil's diagram.

Proof. This result is immediate in view of the 3.4 and 3.5 Theorems.

If we consider a semigroup in which two distinct points a and b are contained in a component, then by letting $A = \{a\}$ it is easy to see that the converse of (3.6) is not true. Moreover, if we look at a totally disconnected space with cardinality > 1 which has the multiplication $xy = x$, it is easy to see that the weaker converse, namely, $A^{(-1)}A = A^{[-1]}A$ implies that A is connected, is also false because in such a semigroup equality holds for any nonempty subset.

CHAPTER IV

ALGEBRAIC RESULTS CONCERNING $\mathcal{H}$-SLICES

To determine whether a subset A in a compact semigroup is an $\mathcal{H}(T)$-slice, for a closed set T, is important because, if so, then A is homeomorphic to a topological group according to the Schutzenberger-Wallace Theorem. One of the results of this chapter reduces the investigation of $\mathcal{H}(T)$-slices in commutative semigroups to those subsets T which are subsemigroups and another result gives necessary and sufficient algebraic conditions for a set A to be an $\mathcal{H}(T)$-slice if T is a subsemigroup. What constitutes such necessary and sufficient topological conditions remains an open question.

It is well known that a semigroup is a group if and only if it is an $\mathcal{H}$-slice so that, in particular, if a semigroup is not a group, then it is not an $\mathcal{H}$-slice (for any $T \subset S$). Moreover, as the following

example indicates, the algebraic conditions on a subset, say A, with cardinality > 1 may be relaxed further and A need not be an $\mathcal{H}$-slice.

<u>4.1 Example</u>. Let S be a semigroup containing more than two elements with multiplication $xy = c$ for some fixed $c \in S$ and let A be any subset of S containing more than one element such that $c \notin A$. Clearly, A is not a semigroup and, since each element is its own $\mathcal{H}$-equivalence class (for any T) in such a semigroup, A is not an $\mathcal{H}$-slice.

The previous example also shows that the conditions $A^{[-1]}A = A^{(-1)}A$ and $AA^{[-1]} = AA^{(-1)}$ are not sufficient for a set A to be an $\mathcal{H}$-slice.

In general, what constitutes necessary and sufficient conditions for a subset of a semigroup to be an $\mathcal{H}$-slice for some T remains an open question; however, if $T^2 \subset T$ we can specify such algebraic conditions as indicated in the subsequent theorem:

<u>4.2 Lemma</u>. Let $T \subset S$ and A be a nonempty subset of S and consider the following conditions:

(1) If a and b are distinct elements of A, then $b \in aT \cap Ta$.

(2) If $a \in A$ and $x \in S\backslash A$, then at least one of the following four sets is empty: $T \cap a^{(-1)}x$, $T \cap xa^{(-1)}$, $T \cap x^{(-1)}a$, $T \cap ax^{(-1)}$.

Then (a) if A is contained in an $\mathcal{H}(T)$-slice, then condition (1) holds.

(b) If condition (1) holds and $T^2 \subset T$, then $A \subset H_a(T)$ for $a \in A$.

(c) If condition (2) holds, then $H_a(T) \subset A$ for $a \in A$.

(d) If $H_a(T) \subset A$ for $a \in A$ and $T^2 \subset T$, then condition (2) is true.

<u>Proof</u>. (a) Suppose $A \subset H_a(T)$ for $a \in A$. If card $A = 1$, then condition (1) is satisfied vacuously; if card $A > 1$, then condition (1) is immediate.

(b) If a and b are distinct elements of A, $T^2 \subset T$, and condition (1) holds, then $L(a) = a \cup Ta = tb \cup Ttb \subset Tb \subset L(b)$ for

some $t \in T$ and, similarly, $L(b) \subset L(a)$ and $R(b) = R(a)$. Thus $H(a) = H(b)$ and, since $T^2 \subset T$, $(a,b) \in \mathcal{H}$.

(c) If $H_a(T) \not\subset A$, i.e., there exists an $x \in S\backslash A \cap H_a(T)$ so that $L(x) = L(a)$ and $R(x) = R(a)$, then the sets in condition (2) are all nonempty.

(d) If condition (2) is not true and $T^2 \subset T$, then for some $x \in S\backslash A$ and for some $a \in A$ all the sets in condition (2) are nonempty and $L(a) = L(x)$ and $R(a) = R(x)$. Thus $H(a) = H(x)$ and, since $T^2 \subset T$, $x \in H_a(T)$ so that $H_a(T) \not\subset A$.

<u>4.3 Theorem.</u> Suppose $T^2 \subset T \subset S$. A nonempty subset A of S is an $\mathcal{H}(T)$-slice if and only if the following conditions hold:

(1) If a and b are distinct elements of A, then $b \in aT \cap Ta$.

(2) If $a \in A$ and $x \in S\backslash A$, then at least one of the following four sets is empty: $T \cap a^{(-1)}x$, $T \cap xa^{(-1)}$, $T \cap x^{(-1)}a$, $T \cap ax^{(-1)}$.

<u>Proof.</u> In view of the lemma, this result is immediate.

If c is an element of a semigroup S such that $xy = c$ for all $x,y \in S$, if T is a subset of S and if $b \neq c$, then $\{b\} = H_b(T)$ and yet there exists no $t \in T$ such that $bt = b$. Hence, this example indicates that the word <u>distinct</u> may not be omitted from condition (1) in the 4.3 Theorem nor may it be removed from condition (1) as it applies in part (a) of (4.2).

If we recall the definitions of the functions la and ra from Chapter I, then it is possible to formulate (4.3) in functional notation:

<u>4.3' Theorem.</u> Let $T^2 \subset T \subset S$. If the domain for the functions la and ra is T, then a nonempty subset A of S is an $\mathcal{H}(T)$-slice if and only if the following two conditions are satisfied:

(1') $la[(la)^{-1}(A\backslash a)] = ra[(ra)^{-1}(A\backslash a)] = A\backslash a$ for each $a \in A$.

(2') If $a \in A$ and $x \in S\backslash A$, then at least one of the following four sets is empty: $(la)^{-1}(x)$, $(ra)^{-1}(x)$, $(lx)^{-1}(a)$, $(rx)^{-1}(a)$.

<u>Proof.</u> It suffices to show the equivalence of the conditions of the 4.3 and 4.3' Theorems. Since $(la)^{-1}(x) = T \cap a^{(-1)}x$, it is

evident that conditions (2) and (2') are the same because in a similar manner equalities for the other three sets may be obtained; and so it remains to exhibit the equivalence of conditions (1) and (1'):

If $la[(la)^{-1}(A\backslash a)] = A\backslash a$ for all $a \in A$ and if b and c are distinct elements of A, then $b \in A\backslash c$ implies the existence of an element $t \in (lc)^{-1}(A\backslash c)$ such that $ct = b$. In a similar manner $ra[(ra)^{-1}(A\backslash a)] = A\backslash a$ for all $a \in A$ implies that $b \in Tc$.

If a and b are distinct elements of A and if $b \in aT$, say $b = at$ for $t \in T$, then $t \in (la)^{-1}(b)$ and $la(t) = b$ so that $A\backslash a \subset la[(la)^{-1}(A\backslash a)]$. Since it is always the case that $la[(la)^{-1}(A\backslash a)] \subset A\backslash a$ and since it is easy to see that, in a similar fashion, $b \in Ta$ implies that $ra[(ra)^{-1}(A\backslash a)] = A\backslash a$ for all $a \in A$, we have that condition (1) implies condition (1').

We conclude this chapter with a result which reduces the study of $\mathcal{X}$-slices in commutative semigroups to those sets T for which $T^2 \subset T$ and $T = \sqcap$. Consequently, (4.3) takes on added significance since it deals with subsets T which are subsemigroups.

4.4 Lemma. If A is such a subset of a semigroup S that card $A > 1$, $A^{[-1]}A = A^{(-1)}A$, condition (1) of (4.2) holds for some subset $T \subset S$ and A is normal in that T, that is, $xA = Ax$ for all $x \in T$, then A is an $\mathcal{H}(T')$-slice where T' is the semigroup generated by $T \cap A^{[-1]}A$.

Proof. For distinct elements $a,b \in A$ we have $[T \cap (ba^{(-1)} \cup a^{(-1)}b)] \subset T'$ so that condition (1) of (4.2) holds when we replace T by T'. Therefore, since T' is a semigroup, it follows from part (b) of (4.2) that $A \subset H_a(T')$ where $a \in A$. Now if $x \in H_a(T')$, then, because $T' \subset A^{[-1]}A$, we have that $x \cup xT' = a \cup aT' \subset A$ and so $H_a(T') \subset A$.

4.5 Theorem. If A is an $\mathcal{H}(T)$-slice which is normal in T and if card $A > 1$, then A is an $\mathcal{H}(T')$-slice where T' is the semigroup generated by $T \cap A^{[-1]}A$. As a result, in a commutative semigroup S to determine if a subset A of cardinality > 1 is an $\mathcal{H}$-slice for some T, it is sufficient to investigate the $\mathcal{H}$-slice decompositions yielded by the subsemigroups of S.

Proof. This is an immediate corollary to (4.4) because the hypothesis that A is an $\mathcal{U}(T)$-slice implies that $A^{[-1]}A = A^{(-1)}A$ and that condition (1) of (4.2) holds.

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Supplementary Readings

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BIOGRAPHICAL SKETCH

Anthony Connors Shershin was born October 16, 1939, at Clifton, New Jersey. Having graduated from Saint Leo Preparatory School at Saint Leo, Florida, in 1957, he then entered Georgetown University in Washington, D.C., and in June, 1961, received the degree of Bachelor of Arts from that institution. He enrolled in the University of Florida in September, 1961, and completed the requirements for the degree of Master of Science in April, 1963. After an absence of one year, Mr. Shershin returned to the University of Florida in September, 1964, and completed the work for the Doctor of Philosophy degree in August, 1967. For both graduate degrees the major was Mathematics and the minor was Physics. During his graduate studies he taught at the University of Florida in the capacities of graduate assistant and part-time interim instructor. From September, 1966, to June, 1967, he taught at the University of Miami in Miami, Florida, while he worked on his doctoral dissertation.

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This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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